B.Sc I
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Problems Based on Resolution into Factors

Important Formulae :→

1. $\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\cdots$

or

$$\sin\theta = \theta\prod_{r=1}^{\infty}\left(1-\frac{\theta^2}{\pi^2 r^2}\right)$$

2. $\cos\theta = \left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\cdots$

or $= \prod_{r=1}^{\infty} 1 - \frac{4\theta^2}{(2r-1)^2\pi^2}$

3. $\sinh\theta = \theta\prod_{r=1}^{\infty}\left(1+\frac{\theta^2}{r^2\pi^2}\right)$

4. $\cosh\theta = \prod_{r=1}^{\infty} 1 + \frac{4\theta^2}{(2r-1)^2\pi^2}$

5. $x^{2n} - 2x^n\cos n\theta + 1$

$$= \prod_{r=0}^{n-1}\left\{x^2 - 2x\cos\left(\theta + \frac{2r\pi}{n}\right) + 1\right.$$

6. (i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty = \frac{\pi^2}{6}$ or $\sum_{r=1}^{\infty}\frac{1}{r^2} = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots \infty = \frac{\pi^2}{8}$

(iii) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots \infty = \frac{\pi^4}{90}$

(iv) $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots \infty = \frac{\pi^4}{96}$

or $\sum_{r=1}^{\infty}\frac{1}{(2r-1)^4} = \frac{\pi^4}{96}$

Q. Resolve $x^{2n} - 2x^n\cos n\theta + 1$ into Factors

Solⁿ :→ Let us first solve the Equation $x^{2n} - 2x^n\cos n\theta + 1 = 0$ —— (1)

It may be written as.

$$(x^n)^2 - 2(x^n)\cos n\theta + (\cos n\theta)^2 + (\sin n\theta)^2 = 0$$

$$\Rightarrow (x^n - \cos n\theta)^2 = -(\sin n\theta)^2 = (i\sin n\theta)^2$$

$$\Rightarrow x^n - \cos n\theta = \pm i\sin n\theta$$

$$\Rightarrow x^n = \cos n\theta \pm i\sin n\theta$$

$$\Rightarrow x = [\cos n\theta \pm i\sin n\theta]^{1/n}$$

$$= \{\cos(2\pi r + n\theta) \pm i\sin(2\pi r + n\theta)\}^{1/n}$$

$$\Rightarrow x = \cos\frac{(2\pi r + n\theta)}{n} \pm i\sin\frac{(2\pi r + n\theta)}{n}$$

Where $r = 0, 1, 2, \ldots, n-2, n-1$

thus the roots of the equation (1) are

$$\cos\theta \pm i\sin\theta,\ \cos\left(\theta + \frac{2\pi}{n}\right) \pm i\sin\left(\theta + \frac{2\pi}{n}\right) - - - -$$
$$- - - \cos\left\{\theta + \frac{2(n-1)\pi}{n}\right\} \pm i\sin\left\{\theta + \frac{2(n-1)\pi}{n}\right\}$$

the factors corresponding to the first pair of roots are

$$(x - \cos\theta - i\sin\theta)(x - \cos\theta + i\sin\theta)$$

i.e $(x - \cos\theta)^2 - i^2\sin^2\theta$

i.e $x - 2x\cos\theta + 1$

Similarly, the second, third - - -, nth pair of the above roots give as quadratic ~~equation~~ factors the Expression

$$x^2 - 2x\cos\left(\theta + \frac{2\pi}{n}\right) + 1$$

$$x^2 - 2x\cos\left(\theta + \frac{4\pi}{n}\right) + 1$$

$$- - - - - -$$

$$x^2 - 2x\cos\left\{\theta + \frac{2(n-1)\pi}{n}\right\} + 1$$

Hence. $x^{2n} - 2x\cos\left\{\theta + \frac{2(n-1)\pi}{n} + 1\right\} = \{x^2 - 2x\cos\theta + 1\}\left\{x^2 - 2x\cos\left(\theta + \frac{2\pi}{n}\right) + 1\right\} - - \cdot \left\{x^2 - 2x\cos\left\{\theta + \frac{2n-2}{n}\pi\right\} + 1\right\}$

i.e

$$x^{2n}-2x^n\cos n\theta+1=\prod_{r=0}^{n-1}\left\{x^2-2x\cos\left(\theta+\frac{2r\pi}{n}\right)+1\right\}$$

* Show that

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\cdots\infty=\frac{\pi^2}{12}$$

$$L.H.S=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\cdots$$

$$=\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\cdots\right)-\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}\cdots\right)$$

$$=\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\cdots\right)-\frac{1}{2^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\cdots\right)$$

$$=\frac{\pi^2}{8}-\frac{1}{4}\frac{\pi^2}{6}$$

$$=\frac{\pi^2}{8}\left(1-\frac{1}{3}\right)$$

$$=\frac{\pi^2}{\cancel{8}4}\times\frac{\cancel{2}}{3}=\frac{\pi^2}{12}$$

* If. 2, 3, 5 .... are all the prime numbers, show that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-1/5^2\right)\cdots=\frac{6}{\pi^2}$$

Solution: $\rightarrow$ L.H.S $=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-1/5^2\right)\cdots$

Since $\left(1-\frac{1}{2^2}\right)^{-1}\left(1-\frac{1}{3^2}\right)^{-1}\left(1-1/5^2\right)^{-1}\cdots$

$$=\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\cdots\right)\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\cdots\right)\left(1+1/5^2+\frac{1}{5^4}+\cdots\right)$$

using $(1-x)^{-1}=1+x+x^2+x^3\cdots$

$$=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}\cdots\cdots$$

$$=\frac{\pi^2}{6}$$

$$\Rightarrow\frac{1}{\left(1-\frac{1}{2^2}\right)}\cdot\frac{1}{(1-3^2)}\cdot\frac{1}{(1-5^2)}\cdots=\frac{\pi^2}{6}\qquad\text{Since } a^{-1}=\frac{1}{a}$$

$$\Rightarrow\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-5^{-2}\right)\cdots=\frac{6}{\pi^2}$$

* Show that

$$\frac{1}{3^4}+\frac{3}{5^4}+\frac{6}{7^4}+\frac{10}{9^4}\cdots=\frac{\pi^2}{64}\left(1-\frac{\pi^2}{12}\right)$$

Solution $\rightarrow$ the Given Expression of L.H.S

$$=\frac{1}{3^4}+\frac{3}{5^4}+\frac{6}{7^4}+\frac{10}{9^4}\cdots$$

$$=\frac{1}{8}\frac{3^2-1}{3^4}+\frac{1}{8}\frac{5^2-1}{5^4}+\frac{1}{8}\frac{7^2-1}{7^4}+\frac{1}{8}\frac{9^2-1}{9^4}\cdots$$

$$=\frac{1}{8}\left[\left(\frac{3^2}{3^4}+\frac{5^2}{5^4}+\frac{7^2}{7^4}+\frac{9^2}{9^4}\cdots\right)-\left(\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots\right)\right]$$

$$=\frac{1}{8}\left[\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}\cdots\infty\right)-\left(\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}\cdots\infty\right)\right]$$

$$=\frac{1}{8}\left[\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\infty\right)-\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}\cdots\infty\right)\right]$$

$$=\frac{1}{8}\left[\frac{\pi^2}{8}-\frac{\pi^4}{96}\right)$$

$$=\frac{\pi^2}{64}\left(1-\frac{\pi^2}{12}\right) \quad \text{proved}$$

Ex:→ Show that

$$\pi/2=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\times\frac{6}{5}\cdot\frac{6}{7}\cdots$$

Solution :→ Since $\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\cdots$

Put $\theta=\pi/2$

$$\Rightarrow \sin\pi/2=\frac{\pi}{2}\left(1-\frac{\pi^2}{4\times\pi^2}\right)\left(1-\frac{\pi^2}{4\times4\pi^2}\right)\left(1-\frac{\pi^2}{4\times3^2\pi^2}\right)\cdots$$

$$\Rightarrow 1=\frac{\pi}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{2^2\times2^2}\right)\left(1-\frac{1}{2^2\times3^2}\right)\cdots$$

$$\Rightarrow 1=\frac{\pi}{2}\left[\left(1+\frac{1}{2}\right)\left(1-\frac{1}{2}\right)\left(1-\frac{1}{2\cdot2}\right)\left(1+\frac{1}{2\cdot2}\right)\left(1-\frac{1}{2\cdot3}\right)\left(1+\frac{1}{2\cdot3}\right)\cdots\right]$$

$$\Rightarrow 1=\frac{\pi}{2}\left(\frac{3}{2}\cdot\frac{1}{2}\ \frac{3}{2}\times\frac{5}{2}\times\frac{5}{4}\times\frac{5}{6}\times\frac{7}{6}\cdots\right.$$

$$\Rightarrow 1=\frac{\pi}{2}\left(\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\times\frac{5}{6}\cdot\frac{7}{6}\cdots\right)$$

$$\Rightarrow \frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\times\frac{6}{5}\times\frac{6}{7}\cdots \quad \text{proved}$$

*

Show
If n be very large
prove that
$$\sqrt{\tfrac{1}{2}(2n+1)\pi} = \frac{2\cdot 4\cdot 6 \cdots 2n}{1\cdot 3\cdot 5 \cdots (2n-1)} \quad \text{Approximately}$$

Solution → As we have
$$\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\cdots\left(1-\frac{\theta^2}{n^2\pi^2}\right)$$
approximately, when n is very large

Putting $\theta = \pi/2 \Rightarrow \frac{\theta}{\pi} = \frac{1}{2}$

we get
$$\sin\pi/2 = \frac{\pi}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{2^2.2^2}\right)\left(1-\frac{1}{3^2.2^2}\right)\cdots\left(1-\frac{1}{n^2.2^2}\right) \quad \text{Approx}$$

$$\Rightarrow 1 = \pi/2\left[\frac{1\cdot 3}{2^2}\cdot\frac{3\cdot 5}{4^2},\frac{5\cdot 7}{6^2}\cdots\frac{(2n-1)(2n+1)}{(2n)^2}\right] \quad \text{Approximately.}$$

$$\Rightarrow 1 = \frac{\pi}{2}\,\frac{1^2, 3^2, 5^2 \cdots (2n-1)(2n+1)}{2^2\,4^2\,6^2\cdots(2n)^2} \quad \text{Approximately}$$

$$\Rightarrow \frac{2^2, 4^2, 6^2 \cdots (2n)^2}{1^2\cdot 3^2\cdot 5^2\cdots(2n-1)^2(2n+1)^2} = \pi/2$$

$$\Rightarrow \frac{2^2, 4^2, 6^2 \cdots (2n)^2}{1^2, 3^2, 5^2\cdots(2n-1)^2} = \frac{\pi}{2}(2n+1)$$

Taking square root
$$\frac{2\times4\times6\times\cdots 2n}{1\cdot 3\times5\cdots 2n-1} = \sqrt{\frac{\pi}{2}(2n+1)}$$

this is also known as
Walli's theorem.

Show that
$$\tan x = \frac{2}{\pi-2x} - \frac{2}{\pi+2x} + \frac{2}{3\pi-2x} - \frac{2}{3\pi+2x} + \frac{2}{5\pi-2x} - \frac{2}{5\pi+2x} \cdots \infty$$

Sol^n →

Since we have

$$\cos x = \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right)\left(1-\frac{4x^2}{5^2\pi^2}\right)\cdots$$

$$= \frac{1}{\pi^2}(\pi+2x)(\pi-2x)\cdot\frac{1}{3^2\pi^2}(3\pi-2x)(3\pi+2x)\cdot\frac{1}{5^2\pi^2}(5\pi+4x)(5\pi-4x)\cdots$$

Taking log ~~of~~ of both sides

$$\log\cos x = \log(\pi-2x)+\log(\pi+2x) - 2\log\pi$$
$$+\log(3\pi-2x)+\log(3\pi+2x)-2\log 3\pi$$
$$+\log(5\pi-2x)+\log(5\pi+2x)-2\log 5\pi\cdots$$

Diff. with respect to x

$$\frac{1}{\cos x}(-\sin x) = \frac{1}{\pi-2x}(-2)+\frac{1}{\pi+2x}(+2)+\frac{1}{3\pi-2x}(-2)+\frac{1}{3\pi+2x}\cdot(2)\cdots$$

$$-\tan x = -\left[\frac{2}{\pi-2x}-\frac{2}{\pi+2x}+\frac{2}{3\pi-2x}-\frac{2}{3\pi+2x}\cdots\right]$$

$$\tan x = \left(\frac{2}{\pi-2x}-\frac{2}{\pi+2x}\right)+\left(\frac{2}{3\pi-2x}-\frac{2}{3\pi+2x}\right)+\left(\frac{2}{5\pi-2x}-\frac{2}{5\pi+2x}\right)\cdots$$

$$= 2\left[\frac{\cancel{\pi}+2x-\cancel{\pi}+2x}{\pi^2-2^2x^2}\right]+2\left[\frac{\cancel{3\pi}+2x-\cancel{3\pi}+2x}{3^2\pi^2-2^2x^2}\right]+2\left[\frac{\cancel{5\pi}+2x-\cancel{5\pi}+2x}{5^2\pi^2-2^2x^2}\right]$$

$$= \frac{8\cdot x}{\pi^2-2^2x^2}+\frac{8x}{3^2\pi^2-2^2x^2}+\frac{8x}{5^2\pi^2-2^2x^2}\cdots$$

$$= 8\sum_{r=1}^{\infty}\frac{x}{(2r-1)^2\pi^2-4x^2}$$

Proved